Abstract—Skills of cutting natural foods are important for robots looking to play a bigger role on kitchen assistance. The basic objective of cutting is to achieve material fracture via smooth movements of a kitchen knife, which in the process performs work to overcome the material toughness, acts against the blade-material friction, and generates shape deformation. During a cutting action, the knife also experiences varying contacts with the material and cutting board. This paper investigates how a robotic arm with force sensing drives the knife to cut through an object in a sequence of three moves: pressing, touching, and slicing. Based on fracture mechanics, position, force, and impedance controls are applied either separately or jointly in the move sequence so the knife follows a prescribed cutting trajectory to split the object. Stabilities of these controls are established. Experiments over several types of fruits and vegetables have exhibited natural cutting movements like would be performed by a human hand.

Index Terms—Modeling of cutting, fracture mechanics, knife pressing, slicing, robot control.

I. INTRODUCTION

AUTOMATION of kitchen skills is an important step towards the advent of multipurpose home robots, which have long been a public fascination. Until today robotic kitchen assistance has been limited to washing and sorting dishes [1], carrying food trays [2], cooking pancakes and noodles [3], making burgers [4], etc. Robotic cooking has been carried out on prepared raw materials in very structured settings [5]. In food factory settings, robots are typically capable of only one task, whether cutting meat, deboning, or butchering chicken. In our life, not so coincidentally, specialized tools are sold at stores or online also for single operations such as slicing lettuce, peeling potatoes, chopping fruits and vegetables, and so on. To play a much bigger role in the kitchen, robots need to be versatile and general purpose — even on just one class of tasks.

Food cutting, as an integral part of automatic meal preparation, stands out as one of the ultimate tests on human-level dexterity for robots. Today, basic cutting skills such as chop, slice, and dice are still out of their reach. Manipulation of soft or irregularly-shaped food items aside, one main technical challenge for robotic cutting is how to plan and control a knife’s movement through a material while reacting to encountered forces of different natures (fracture, friction, viscosity, and contact) by the material and cutting board. Such control needs to utilize some knowledge about these forces as well as shape deformations, which can be modeled based on elasticity theories [6], [7] and fracture mechanics [8].

Although the goal of cutting is steady progress leading up to a complete separation of the material, changing contacts and path constraints divide the action into periods which bear specific control objectives. In one period, for example, the knife needs to progress on material fracture. In another period, the knife needs to come to a stop as soon as its edge touches the cutting board.

This paper investigates how a robotic arm moves a kitchen knife to cut through objects smoothly. The considered objects include vegetables such as potatoes and fruits such as apples that will undergo negligible deformations during the cutting process. The work to be presented bears a number of characteristics below.

1) It investigates an under-researched form of manipulation which alters the structure of the manipulated object in the process.
2) It studies a kitchen knife skill via a decomposition into three phases (illustrated in Fig. 1): pressing, in which the knife moves downward along a prescribed trajectory until its edges make contact with the cutting board; touching, in which the knife softens its impact with the cutting board; and slicing, in which the knife separates the object completely with its edge sliding on the board across the object’s bottom.
3) Each of the three phases above, with its own objective and contact constraints, is carried out under a separate control strategy, whether on position, position/impedance, or position/force.
4) These control strategies, with stability analyses, make use of the fracture and frictional forces...
modeled based on fracture mechanics.

5) Cutting experiments are performed on natural fruits and vegetables (rather than artificial objects as often studied in fracture mechanics).

The rest of the paper is organized as follows. Section II reviews works in related areas including fracture mechanics, cutting of soft tissues, and robot control. Section III characterizes the technical problem and presents its underlying mechanics. In Section IV, cutting is divided into the aforementioned three phases, for each of which a separate control strategy is devised to adjust to the changing contact situation. Section V describes the results from cutting four types of fruits/vegetables with a WAM Arm utilizing its two degrees of freedom (DOFs) in a vertical plane. Section VI extends the cutting strategy to a robotic arm with more DOFs in the plane. Summary, discussion, and future work follow in Section VII.

This paper has extended an earlier conference version [9] in several aspects. The knife orientation is now controlled directly in the pressing phase, rather than exerted as a constraint, to allow more flexibility of cutting and also simplify control. Impedance control is added to lessen the impact between the knife and cutting board. Stabilities are established for all the control strategies. The paper also extends the cutting scheme to a robotic arm with more than two DOFs in the cutting plane, and includes extensive experiments for validation purpose.

II. RELATED WORK

Fracture mechanics [10] builds on a balance between the work done by cutting and the total amount spent for crack propagation, transformed into other energy forms (strain, kinetic, chemical, etc.), and dissipated by friction. Methods for measuring the cutting force and fracture toughness were studied for ductile materials [11] and live tissues [12], [13]. A “slice-push ratio” was introduced in [14] to quantify the works done along two orthogonal directions, and then to formulate the dramatic decrease in the fracture force when the knife was simultaneously pressing and slicing the material. A different explanation [15] for such decrease stated that pushing caused global deformation while slicing yielded local deformation (and thus required less effort to create fracture). The fracture force and torque could be obtained via an integration along the edge of the blade [16]. In our paper, this approach has been extended to account for blade-material friction in modeling.

Stress and fracture force analyses, supported by simulation and experiment, were performed for robotic cutting of biological materials, accounting for factors such as blade sharpness and slicing angle [17]–[19]. In surgical training, realistic haptic display of soft tissue cutting is quite important. Haptic models were developed for animal tissue cutting with a scissor [20], soft tissue deformation prior to fracture [21], as well as needle insertion into soft tissues [22]. Most of these models, however, tended to be empirical. We refer to [23] for a survey on mechanics and modeling of cutting biological materials.

The knife carries out cutting along some trajectory through contacts between its blade with the material and the cutting board. While position control [24, pp. 190-199] realizes trajectory following, force control [25] robustly deals with modeling and execution errors in contact tasks. Since contacts experienced by the knife vary during a complete cutting action, it is natural to employ multiple control policies. Hybrid force/position control [26], for instance, is a natural choice when the knife is slicing through an object while maintaining
contact with the cutting board. Impedance control [27], which adjusts contact force from a motion deviation like an intended mass-spring-damper, can be employed during fast cutting to reduce the impact between the knife and cutting board. For the entire action of cutting to look natural, smooth transitions among these policies would be desirable. Switching between position and force controls was shown to regulate the contact force during an impact and realize a smooth contact transition [28].

To deal with contact constraints, controls of force and position are more effectively conducted in the workspace [29] using a reduced set of coordinates [30, pp. 501-510]. Keeping the knife’s orientation during a period of cutting, sometimes desirable, can also be carried out as control of a constrained manipulator [24, pp. 202-203].

Robotic cutting has been investigated in a number of ways: adaptive control based on position and velocity history to learn the applied force [31], adaptive force tracking via impedance control [32], visual servoing coupled with force control [33], and impedance control for cooperation of a cutting robot and a pulling robot [34]. Adaptive impedance control was carried out to minimize force error in cutting a nonhomogeneous workpiece [32], utilizing a bound for the gain yielded from stability analysis. A 2-DOF robot [35] demonstrated how to debone a bird by following a cutting path determined from x-ray imaging based on force feedback, with the help of a passive mechanism for fixation.

Data driven robotic cutting has also been investigated. In [36], a generative model for representing objects’ properties was created from collected haptic information, and predefined actions were chosen based on the model. Learning algorithms were proposed in [37] to estimate the optimal input force for control during cutting.

III. MECHANICS OF CUTTING

A. Notation

In this paper, a vector is represented by a lowercase letter in bold, e.g., \( \mathbf{a} = (a_x, a_y)^T \), with its \( x \)- and \( y \)-coordinates denoted by the same (non-bold) letter with subscripts \( x \) and \( y \), respectively. A unit vector has a hat, e.g., \( \hat{a} = a / \|a\| \). The cross product \( a \times b \) of two vectors \( a \) and \( b \) is treated as a scalar. The subscripts \( d \) and \( e \) refer to the desired value and error, respectively. For example, \( a_{yd} \) and \( a_{ye} \) are the desired value and error for \( a_y \). A matrix is denoted by an upper case letter, e.g., \( A \), and its pseudo-inverse has the superscript \(^+\), e.g., \( A^+ \). An \( n \times n \) identity matrix is denoted by \( I_n \). A superscript in the form of a parenthesized number refers to the expression of the corresponding variable as given in the equation referenced by that number, e.g., \( \tau_a^{(32)} \) refers the expression of \( \tau_a \) in (32). Table I summarizes the notation used in this paper.

B. Assumptions

As shown in Fig. 2, cutting of an object takes place in the vertical \( x\)-\( y \) plane, referred to as the world frame, located at some point \( o \) on the cutting board. The blade of a knife is often quite thin. To have a clean presentation, we make the following assumption about the knife used for cutting:

(A1) The knife’s blade has negligible thickness.

The knife is considered very sharp, which makes the next assumption reasonable:

(A2) Contact friction between the knife’s edge and cutting board is negligible.

Under assumption A2, when the knife’s edge touches the board, it receives a vertically upward contact force. This will facilitate control of slicing to be presented in Section IV-C.

Our next assumption is about the object:
Differentiation with respect to time.

(A3) The object remains stable during cutting with negligible dynamic effects.
The object is stabilized on its own (e.g., lying on a flat base) or by some fixture. Dynamic effects are negligible because cutting proceeds at a relatively slow speed.

Some foods such as potatoes and yams barely deform during cutting. In this paper, we will relieve ourselves from deformable modeling with the last assumption:

(A4) The material being cut has negligible deformation.

C. Task Geometry

The knife in Fig. 2 is rigidly attached to the open end \( \alpha \) of a 2-DOF robotic arm, whose base is located at \( b \) and whose two links of lengths \( l_1 \) and \( l_2 \) move in the \( x-y \) plane. The corresponding two joint angles are denoted by \( \theta_1 \) and \( \theta_2 \). At \( \alpha \) is attached a frame \( x''-y'' \) referred to as the arm frame. Writing

\[
\theta = (\theta_1, \theta_2)^T, \\
\hat{l}_1 = \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \end{pmatrix}, \\
\hat{l}_2 = \begin{pmatrix} \cos(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) \end{pmatrix},
\]

we obtain the arm frame’s position

\[
\alpha = \begin{pmatrix} a_x \\ a_y \end{pmatrix} = b + l_1 \hat{l}_1 + l_2 \hat{l}_2, \quad (3)
\]

and orientation

\[
\phi = \theta_1 + \theta_2. \quad (4)
\]

Attached to the knife point \( p \) is a local frame \( x'-y' \) (see Fig. 2), called the knife frame. This frame, rigidly connected to the arm frame \( x''-y'' \), rotates from the latter through a constant angle \( \psi'' \), and thus, from the world frame through an angle

\[
\psi = \theta_1 + \theta_2 + \psi''. \quad (5)
\]

The shapes of kitchen knives differ by culture. Some have straight edges and spines, and some have curved ones. The kitchen knife considered here has both curved...
edge and spine, in part because knives of this type are quite common, and in part because straight edge and spine can be considered as special cases of curved ones. As depicted in Fig. 3, the knife’s edge and spine are described in its own $x'$-$y'$ frame by two curves $\beta'(u) = (\beta'_x, \beta'_y)'$ and $\gamma'(q) = (\gamma'_x, \gamma'_y)'$, respectively, such that $\beta'(0)$ and $\gamma'(0)$ coincide with the frame’s origin at the knife point $p$. In the world frame $x$-$y$, they are thus described by the following two curves:

$$\beta(u) = \begin{pmatrix} \beta_x \\ \beta_y \end{pmatrix} = p + R(\psi)\beta'(u), \quad (6)$$

$$\gamma(q) = \begin{pmatrix} \gamma_x \\ \gamma_y \end{pmatrix} = p + R(\psi)\gamma'(q),$$

where

$$R(\psi) = \begin{pmatrix} \cos \psi & -\sin \psi \\ \sin \psi & \cos \psi \end{pmatrix}$$

is the rotation matrix for the knife.

As cutting proceeds, the edge intersects the object at a section of $\beta(u)$ over some interval $[u_1, u_2]$, $u_1 \leq u_2$. That $u_1 = u_2$ holds at the start (time $t = 0$) and end of cutting. The section is denoted $\beta[u_1, u_2]$ for convenience. A section $\beta[u_1, u_2]$ of the spine $\gamma(q)$ over $[q_1, q_2]$, for some $q_1$ and $q_2$, may also be inside the object. Both curve sections are illustrated in Fig. 3.

Since the object does not deform under Assumption A4, we let the curve $\sigma'(r)$ describe its non-varying contour of the cross section intersected by the $x$-$y$ plane. The knife’s edge intersects the curve at $\sigma(r_1)$ and $\sigma(r_2)$ from left to right. Clearly,

$$\beta(u_1) = \sigma(r_1), \quad \beta(u_2) = \sigma(r_2).$$

The segments $\beta[u_1, u_2]$ and $\sigma[r_2, r_1]$ enclose the fracture region $\Phi$ (see Fig. 3). When a section of the spine $\gamma[q_1, q_2]$ is inside the cross section, it is bounded by $\sigma[r_4]$ and $\sigma[r_3]$ such that

$$\gamma(q_1) = \sigma(r_4), \quad \gamma(q_2) = \sigma(r_3).$$

The four segments $\beta[u_1, u_2], \sigma[r_2, r_1], \gamma[q_1, q_2]$, and $\sigma[r_4, r_1]$ bound the contact region $\Omega$. Clearly, $\Omega \subseteq \Phi$.

### D. Forces During Cutting

During cutting, let $v_p$ be the velocity of the knife point $p$, and $\omega$ the knife’s angular velocity. The edge segment $\beta[u_1, u_2]$ experiences a force $f_C$ due to material fracture. In other words, the work done by $-f_C$ is yielding new fracture.\(^1\) Consider an infinitesimal element

\[^1\]On a deformable object, the knife would also exert a force $-f_U$ which causes an increase (or decrease) in the object’s strain energy.

of length $ds$ on the knife’s edge starting at $u \in [u_1, u_2]$. See Fig. 4. The element may or may not be generating fracture under the knife’s rotation. We need only consider the former case here. The element exerts the force $-df_C$ in the direction of its velocity

$$\nu = v_p + \omega R(\psi) \begin{pmatrix} -\beta'_y \\ \beta'_x \end{pmatrix},$$

and for a movement of distance $dn$, generates an area of fracture that is a parallelogram (shown in Fig. 4). Its four sides are parallel to either the edge tangent

$$i = \left(\frac{d\beta_x}{du}, \frac{d\beta_y}{du}\right) \div \left(\frac{d\beta_x}{du}, \frac{d\beta_y}{du}\right),$$

or the velocity $\nu$. The material’s fracture toughness $\kappa$ is defined to be the energy required to propagate a crack by unit area [10, p. 16]. We have

$$(-df_C \cdot \dot{\nu})dn = -\kappa(\dot{\nu} \cdot \hat{n})dn ds,$$

where $\hat{n}$ is the unit inward normal at $\beta(u)$, $\dot{\nu} = \nu/\|\nu\|$, and

$$df_C = \kappa(\dot{\nu} \cdot \hat{n}) \dot{\nu} ds$$

$$= \kappa \left(\dot{\nu} \cdot \left(-\frac{d\beta_x}{du}, \frac{d\beta_y}{du}\right)^T\right) \dot{\nu} du.$$

Integration over the segment $S = \beta[u_1, u_2]$ yields the total fracture force:

$$f_C = \int_S df_C. \quad (7)$$

Since the knife is rigidly attached to the robotic arm’s open end $a$, the fracture force yields a torque at the point:

$$\tau_C = \int_S (\beta(u) - a) \times df_C. \quad (8)$$

Coulomb friction exists in the contact region $\Omega$ on both sides of the blade. Denote by $f_f$ the frictional force exerted on the knife. Let $P$ be the pressure distribution and $\mu$ the coefficient of friction. Let the unit vector $\hat{v}(x, y)$ donate the direction of the velocity of an area

![Fig. 4. Area of fracture yielded by an element of length $ds$ on the knife’s edge.](image-url)
element at \((x, y)^T \in \Omega\). The force and torque at the open end \(a\) due to friction are given below:

\[
\begin{align*}
f_F & = -2\mu p \int \int_\Omega \hat{\nu} \, dx \, dy, \\
\tau_F & = -2\mu p \int \int_\Omega \left( \left( \frac{x}{y} \right) - a \right) \times \hat{\nu} \, dx \, dy.
\end{align*}
\]

(9) (10)

The wrenches \((f_C, \tau_C)\) and \((f_F, \tau_F)\) will be used for knife control during touching and slicing in Sections IV-B and IV-C later. They can be evaluated given the knife’s pose \((p, \psi)\) and velocities \((v, \omega)\). If the knife is translating, they have simple forms that are derived in Appendix A. In the general case, the velocities of the points on the knife edge and inside the contact area \(\Omega\) vary, which implies that the force and frictional forces and torques can only be calculated numerically.

Besides causing fracture and overcoming friction, the arm needs to balance the knife’s gravitational force and its resulting torque. The wrench (force and torque) exerted at the arm’s open end \(a\) due to cutting, friction, and knife gravity is

\[
\rho_a = \left( f_C + f_F - mg \hat{y} \right) \text{and} \tau_a = \left( \tau_C + \tau_F - mg (c_m - a) \times \hat{y} \right),
\]

where \(m\) is the knife’s mass, \(c_m\) the location of its center of mass, \(g > 0\) the gravitational acceleration, and \(\hat{y} = (0, 1)^T\). The wrench is measured by a force/torque \((F/T)\) sensor mounted at \(a\) (whose location is effectively extended by the sensor).

IV. DYNAMICS AND CONTROL OF CUTTING

Cutting of an object proceeds in three phases that were previously illustrated in Fig.1. The first phase is pressing, during which the arm translates the knife downward until its edge contacts with the cutting board. The second (transitional) phase is touching during which the arm reduces the magnitude of knife’s vertical velocity to soften the robot-knife contact. The third phase is slicing during which the arm translates and rotates the knife to move its contact point with the cutting board across the object’s bottom segment \(\overline{PR}\) in the cutting plane. By now the object has been split into two parts.

A. Pressing

The relative position and orientation of the arm frame to the world frame is described by the vector \(x = (\alpha^T, \phi)^T\), with \(\alpha\) and \(\phi\) given in (3) and (4), respectively. Immediately,

\[
\dot{x} = J_a \dot{\theta},
\]

where \(J_a\) is the \(3 \times 2\) Jacobian at \(a\):

\[
J_a = \frac{\partial}{\partial \theta} \left(b + l_1 \hat{l}_1 + l_2 \hat{l}_2 \right).
\]

In the pressing phase, the arm dynamics are as follows:

\[
J_a^T \rho_a + \tau = M(\theta) \ddot{\theta} + C(\theta, \dot{\theta}) \dot{\theta} + \tau_g(\theta),
\]

(12)

where \(\tau\) is the joint torque vector, \(M(\theta)\) is the arm’s \(2 \times 2\) mass matrix, \(C(\theta, \dot{\theta}) \dot{\theta}\) includes the Coriolis and centrifugal terms, and \(\tau_g(\theta)\) is the gravity term.

Since the robot has 2 DOFs, only two of the three pose variables \(a_x, a_y\), and \(\phi\) are independent. We choose \(a_x\) and \(a_y\) for the convenience of specifying a cutting path. Particularly, \(a_y\) reflects the knife’s downward movement, which affects the cutting rate directly. We make use of \(\dot{a} = \dot{J}_a \dot{\theta}\), where

\[
\dot{J}_a = \left( \frac{\partial a_x}{\partial \theta}, \frac{\partial a_y}{\partial \theta} \right)^T
\]

is a submatrix of \(J_a\), and rewrite the dynamics in the Cartesian coordinates as

\[
\tau = M \dot{J}_a^{-1} \dot{a} + \tau_a.
\]

(13)

In the above,

\[
\tau_a = (C - M \dot{J}_a^{-1} \dot{J}_a) \dot{\theta} + \tau_g(\theta) - J_a^T \rho_a
\]

(14)

includes the Coriolis, centrifugal, and gravity effects, and the external wrench \(\rho_a\).

For a desired cutting path \(a_d = (a_{xd}, a_{yd})^T\), the following position controller is proposed:

\[
\tau = M \dot{J}_a^{-1} \left( \frac{\lambda_x}{\lambda_y} \right) + \tau_a,
\]

(15)

where

\[
\left( \frac{\lambda_x}{\lambda_y} \right) = \dot{a}_d + K_{vl} \dot{a}_e + K_{pl} a_e + K_{vl} \int a_e \, dt
\]

(16)

with \(K_{vl}, K_{pl}\), and \(K_{vl}\) respectively being some proportional, integral, and derivative (PID) \(2 \times 2\) gain matrices. To simplify the controller, they are set as diagonal matrices.

At time \(t = 0\), the knife’s edge makes contact with the object at \(a_{yd}(0) = \gamma_0\), which can be determined from the initial configuration of the knife and the contour of the object in the cutting plane. The pressing phase ends when contact between the knife and the cutting board is detected from a sudden increase in the force reading by the \(F/T\) sensor. At that moment, say, \(t_1\), the lowest point on the knife’s edge is slightly into the cutting board. Over \([0, t_1]\) the desired velocity \(\dot{a}_{yd}(t)\) first decreases, i.e., the
speed $|\dot{\alpha}_{pd}(t)|$ increases, and then stays at a constant value.

During pressing, the desired path is often chosen to be one with constant end-effector orientation so the knife is in translation, in order to simplify the calculation of fracture and frictional forces.

B. Touching

The initial point of contact in the touching phase

$$c = \begin{pmatrix} c_x \\ c_y \end{pmatrix} = p + R(\psi)\beta'(u)$$

(17)

needs to start to the left endpoint $p_l$ of the object’s bottom segment (see Fig. 5). This can be done by setting the desired trajectory $\alpha_d$ properly.

The upward contact force is represented by a scalar $f_y$. This force is a component of the total force $f_S$ received at the open end $a$, which is part of $p_n$ from the reading by the F/T sensor mounted at $a$, that is

$$f_S = f_C + f_F + (f_y - mg)\ddot{y}.$$  

(18)

Here, the fracture force $f_C$ and the friction force $f_F$ are obtained from modeling described in Section III. Therefore, we can estimate $f_y$ using the above equation to control it later.

Under the knife-board contact, the dynamics has changed from (12) to

$$\tau = M\ddot{\theta} + C\dot{\theta} + \tau_g - J_a^T\rho - J_c^T\begin{pmatrix} 0 \\ f_y \end{pmatrix},$$

(19)

where, from (17),

$$J_c = \frac{\partial c}{\partial \theta} = \frac{\partial p}{\partial \theta} + \frac{\partial R(\psi)}{\partial \theta}\beta'(u)$$

(20)

($u$ treated as a constant) is the Jacobian of $c$. Differentiating $\dot{a} = J_a\dot{\theta}$ and substituting the obtained $\dot{\theta} = J_a^{-1}(\ddot{a} - \dot{J}_a\dot{\theta})$ into (19), we transform it to

$$\tau = M\ddot{\theta} + \tau_a - J_c^T\begin{pmatrix} 0 \\ f_y \end{pmatrix}.$$  

(21)

A large contact force could halt the arm or even cause some damage to it, so the robot has a need to decelerate as soon as contact with the cutting board is detected.

Impedance control in the following form can achieve the purpose:

$$\tau = M\ddot{\theta} + J_a^{-1}\lambda_x + \tau_a - J_c^T\begin{pmatrix} 0 \\ f_y \end{pmatrix},$$

(22)

where the servo $\lambda_x$ in the $x$-direction is given in (16) with zero desired velocity and acceleration, while the servo in the $y$-direction is

$$\lambda_y' = \dot{a}_{yd} + \frac{k_r a_{ye} + d_r a_{ye} + f_y}{b_r},$$

where $k_r$, $d_r$, and $b_r$ are some stiffness, damping, and inertia values for a desired impedance behavior, respectively.

The reason for using impedance control rather than force control is that the former handles velocity to make its behavior similar to a damper. Force control, on the other hand, does not quite perform as adaptively to the impact velocity.

Impedance control ends when the contact force varies little after a significant decrease. Since this type of control is known not for precise force regulation, further adjustment of the contact force will follow.

C. Slicing

Slicing starts right after touching. During this phase, the knife moves on the cutting board to split the object.

To ensure separation, the knife-board contact force is maintained at some desired level throughout the phase. As illustrated in Fig. 6, the arm frame $x''$-$y''$ rotates in order to keep the knife-board contact, which as a result moves on both the knife’s edge (located by $\beta(u)$) and the board (located by $c$). The knife undergoes a changing rotation of the angle $\psi$ given in (5) from the world frame. The knife point is at

$$p = a + R(\psi)p',$$

where $p'$ is its (fixed) position in the arm frame. It follows from (6) that the moving edge curve $\beta$ depends...
on $\psi$, and therefore on the joint angles $\theta$. The contact point $\beta(u)$ on the curve is at $c = (c_x, 0)^T$. Thus,

$$
\begin{align*}
\beta_x &= c_x, \quad (23) \\
\beta_y &= 0. \quad (24)
\end{align*}
$$

The contact point $c$ is the lowest point on $\beta$, satisfying

$$
\frac{\partial \beta_y}{\partial u} = 0. \quad (25)
$$

Equations (23)–(25) define $\theta_1, \theta_2,$ and $u$ as functions of $c_x^2$. Given $c_x$, we can solve for $\theta_1, \theta_2,$ and $u$ using Newton’s method or the homotopy continuation method [38]. This prompts us to replace $\theta$ with $c_x$, since the rest of cutting reduces to moving the point $c$ from $p_l$ to $p_r$ along the $x$-axis. The Jacobian for the above coordinate transformation is

$$
\iota_c = \frac{d\theta}{dc_x}, \quad (26)
$$

so that

$$
\dot{\theta} = \iota_c \dot{c}_x. \quad (27)
$$

The derivative $\iota_c$, along with $du/dc_x$, can be solved from the three linear equations generated from differentiating (23)–(25) with respect to $c_x$. For details, we refer to Appendix B.

The motion of the knife during slicing is restricted by its contact with the cutting board. The dynamics given in (21), after the substitution $\ddot{\theta} = \iota_c \ddot{c}_x + \iota_c \dot{c}_x$ from differentiating (27), are transformed into

$$
\tau = M \iota_c \ddot{c}_x + \iota_c \dot{c}_x - J_a^T \left(0 \ f_y\right), \quad (28)
$$

where

$$
\iota_c = M \iota_c \ddot{c}_x + C \dot{\theta} + \tau_g - J_a^T \rho_a. \quad (29)
$$

The knife-board contact force $f_y$ is estimated from modeling and the reading by the F/T sensor using (18). The derivative $\dot{\iota}_c$ can be evaluated from differentiating (23)–(25), which is also detailed in Appendix B.

Let $c_{xd}(t)$ be some desired time trajectory of $c_x$ and $c_{xe}(t) = c_{xd} - c_x$ be the position error of the contact point $c$. The desired constant normal force exerted on the table is $-f_d$. The force error is $f_e = f_d - f_y$. We now apply a third control law below:

$$
\tau = M \iota_c \left(\dot{c}_{xd} + k_{i,c} c_{xe} + k_{p,c} c_{x\dot{e}} + k_{i,c} \int c_{xe} \, dt\right) + \tau_c - J_a^T \left(f_d + k_{fi} \int f_e \, dt\right),
$$

where $k_{i,c}, k_{p,c},$ and $k_{fi}$ are the PID gains, respectively, and $k_{fi}$ is the integral gain for regulating the contact force.

D. Control Architecture and Stabilities

Fig. 7 (a) shows the basic system diagram, in which the controller handles all the position and force inputs from specifications, the kinematics of the arm, F/T sensor, and modeling. Part (b) of the figure details hybrid
position/force control in the slicing phase. Stabilities of the controllers (15), (22), and (30) applied in the three cutting phases are established in Appendix C.

V. EXPERIMENTS

![Experimental setup](image)

**TABLE II**

<table>
<thead>
<tr>
<th>Objects</th>
<th>$\mu$</th>
<th>$\kappa$ (N/m)</th>
<th>$P$ (N/m²)</th>
</tr>
</thead>
<tbody>
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<td>675</td>
<td>2927</td>
</tr>
<tr>
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<td>270</td>
<td>2000</td>
</tr>
<tr>
<td><img src="image" alt="Image" /></td>
<td>0.6</td>
<td>400</td>
<td>2700</td>
</tr>
<tr>
<td><img src="image" alt="Image" /></td>
<td>0.7</td>
<td>370</td>
<td>2500</td>
</tr>
</tbody>
</table>

Shown in Fig. 8, the arm used for cutting was a 4-DOF Whole Arm Manipulator (WAM) from Barrett Technology, LLC. Its joints 1 and 3 were fixed so the robot effectively had two DOFs. We derived the arm’s dynamics equation (12) according to its specifications [39].

Mounted on the end effector was a 6-axis Delta IP65 F/T sensor from ATI Industrial Automation. Its geometry, mass, and inertial properties are available. A Microsoft Kinect sensor was used in acquiring some densely distributed points on the object’s surface. Those points close enough to the cutting plane were fit over to construct the contour $\sigma(r)$ of the cross section, where $r$ is the $x$-coordinate in the world frame. The kitchen knife was rigidly mounted on the F/T sensor using a metal adaptor, so its kinematics were in terms of the robot’s joint angles. Before cutting, the robotic arm moved to a fix position where the knife was not in contact with the object.

A 4-DOF Servo Motor Arm was used to stabilize the object during cutting. The small arm could also repeatedly push the uncut portion of the object forward on the cutting board for a specified distance.

To model a kitchen knife, we placed it on a sheet of paper, and drew its contour. The knife’s edge $\beta'(u)$ was reconstructed in the knife frame $x'-y'$ as follows. Setting $u$ to be the $x'$-coordinate, we fit a quadratic curve $\beta'_y$ to the $y'$-coordinates of the measured points on the edge. Similarly, the knife’s spine was reconstructed through fitting as a quadratic curve $\gamma'(q) = (q, \gamma'_y(q))^T$ with $q$ identified with $x'$.

To measure the coefficient of friction between the blade and a material, we cut an object of this material in half by hand and let one resulting piece rest with its newly cut face on the knife’s blade. Then we tilted the blade until the piece began to slide. The tangent of slope angle for the blade at this moment was used as the estimate.

To measure the fracture toughness and pressure distribution, the robot drove the knife to cut into an object of the same type. During this process, the F/T sensor obtained a sequence of readings, each accounted for both the fracture and frictional forces. Next, the arm moved the knife upward until it was above the piece and then moved it downward into the crack for a second time. The new sequence of readings produced during this descent accounted for the frictional force only. Readings from the two sequences had a correspondence by the height of the knife. Subtraction of the readings in the second sequence from the corresponding ones in the first sequence recovered fracture force values during the cutting action. The work done by the knife during cutting was thus estimated, so was the material’s fracture toughness via division by the measured fracture area. The pressure distribution was estimated, during the knife’s second descent, from the frictional force readings...
TABLE III
CONTROL GAINS FOR INDIVIDUAL PHASES OF CUTTING IN THE EXPERIMENTS.

<table>
<thead>
<tr>
<th>Pressing</th>
<th>Touching</th>
<th>Slicing</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_{p[a]}$</td>
<td>$K_{i[a]}$</td>
<td>$K_{v[a]}$</td>
</tr>
<tr>
<td>500</td>
<td>800</td>
<td>35</td>
</tr>
</tbody>
</table>

and the areas of the blade inside the object at the same time instants.

Onions, potatoes, apples, and cucumbers were used in the experiment. Table II lists the measured values of the physical parameters for these four types of objects. These parameter values could vary with freshness of the
food item and the thickness of its parts on both sides of the blade.

In a cutting trial, half of an object (precut by the human hand) was placed on a table with its flat face down (see the potato in Fig. 8).

Values of the control gains for all the three phases of cutting are listed in Table III. Applied to all four object types, these gains were chosen to keep the robotic arm stable and from generating excessive torque responses at its joints. Such an undesired situation could occur, for instance, with impedance control during touching if a very large value for the gain $d_r$ is chosen, commanding torques to exceed the limits for the arm.

A. Phase-by-Phase Control Validation

Cutting starts with the knife making an initial contact with the object. The arm’s joint angles can be calculated from a preselected knife orientation. The three phases of pressing, touching, and slicing are carried out over the time periods $[0, t_1]$, $[t_1, t_2]$, and $[t_2, t_3]$, respectively.

Pressing. Fig. 9 presents the data obtained from cutting an onion. The pressing phase ended with an error of 0.8 mm in the open end’s $x$-position, an error of 1.3 mm in its $y$-position and a deviation of 0.002 rad in its orientation (see the plots in (b)-(d)). The plot in (e) shows that the sensed force $f_S$ and the sum of the modeled fracture force $f_C$ and frictional force $f_F$ has an average discrepancy around 8 N in the $y$ direction. Their discrepancy in the $x$ direction as shown in (f), is around 2 N on the average. These discrepancies were mainly due to two factors. One was the inaccurate contour $\sigma(r)$ of the cross section, which was obtained by fitting over points acquired by the Kinect sensor. The other factor was that the three parameters ($\mu$, $\kappa$, and $P$) varied slightly from one object to another of the same type.

Touching. This phase was detected from a sudden increase in the $y$-component of the force reading as triggered by the knife’s contact with the cutting board. Fig. 9(g) shows a snapshot from this second phase of the same cutting action. As plotted in Fig. 9(h), the knife-board contact force under impedance control stopped the increase quickly and then started to decrease. Since the knife barely moved, the fracture and frictional forces were negligible so the knife-board contact force accounted for the $y$-component of the sensor reading. The touching phase ended after this contact force had become stabilized.

Slicing. A snapshot of the following slicing phase of the action is shown in Fig. 9(i). The knife was translating and rotating, which resulted in a movement by the contact point for a distance (i.e., change in $s$) that was neither zero nor equal to that (i.e., change in $c_x$) of its movement on the cutting board. This is illustrated by an increasing gap between the $s$ and $c_x$ trajectories in (j). As shown in (k), the contact force component $f_y$ was maintained close to the desired value. This component was estimated by subtracting from the sensor reading the modeled fracture and knife-material friction forces (which were close to 0). The end of slicing was detected visually after the knife-board contact point moved out of the object.

B. Complete and Repeated Cutting

Fig. 10(a)–(e) shows the experimental results from cutting a potato. Included in (a) are four snapshots respectively at the start and during the three phases. Here we let $c = (c_x, c_y)^T$ also denote the lowest point on the knife’s edge during the pressing and touching phases. In (b), the ordinate $c_y$ follows the desired trajectory $c_{yd}$ (which was obtained from $a_{yd}$, since the knife was rigidly connected to the robot) during pressing, and the abscissa $c_x$ follows the desired trajectory $c_{xd}$ throughout the three phases of cutting. Fig. 10(c), plotted over the period $[t_1, t_3]$, shows that the contact force decreased quickly under impedance control during touching, and then converged to the desired value with small variations under force control during slicing. In the pressing phase, as shown in (d), the modeled frictional force $f_F$ (between the blade and the material) and fracture force $f_C$ add up close to the sensed force $f_S$. The same plot also shows that, during slicing, the $y$-component of the modeled force, $f_F + f_C$ was very small, since every point on the knife edge was moving close to the $x$-direction. Also shown in (d), the force reading at the start of cutting was not zero in $x$ or $y$ directions. This was due to an inaccurate contour reconstructed from points obtained from the Kinect sensor so the knife had been pressing against the object already before cutting started. In (e), the orientation $\theta_1 + \theta_2$ of the arm frame (hence that of the knife) stays almost constant in the first two phases, and increases in the third phase since the knife had to rotate to maintain its contact with the cutting board.

Convergence to the desired knife motions and force magnitudes were also observed in apple cutting of which an instance is shown in Fig. 10(f)–(h).

The robotic arm could perform cutting actions in a sequence. Fig. 11 includes some snapshots from cutting a cucumber into pieces.

VI. Cutting With Higher Degrees of Freedom

The three-phase cutting strategy presented in Section IV can be extended for a robotic arm with $n > 2$
revolute joints which have parallel horizontal axes (so their driven links can be viewed as moving in the same vertical plane). The extra DOFs make it possible for the arm to maintain the knife’s orientation during the third phase of slicing. The unit link directions \( \hat{l}_i \) and \( l_2 \) defined in (1) and (2) now assume the forms \( \hat{l}_1 = (\cos(\sum_{j=1}^{i} \theta_j), \sin(\sum_{j=1}^{i} \theta_j)) \), \( i = 1, \ldots, n. \)

The arm frame has the orientation \( \phi = \sum_{j=1}^{n} \theta_j \) and the Jacobian \( J_a = \frac{\partial x}{\partial \theta} \) is now a \( 3 \times n \) matrix. Denote by \( J_a^+ \) the pseudo-inverse of \( J_a \). Except for some degenerate arm configurations, the rows of \( J_a \) are linearly independent and \( J_a^+ = J_a^T(J_a J_a^T)^{-1} \), so that

\[
J_a J_a^+ = I_3. \tag{31}
\]

Redefine \( \tau_a \) introduced in (14) as

\[
\tau_a = (C - MJ_a^+ J_a \dot{\theta}) \dot{\theta} + \tau_g - J_a^T \rho_a \tag{32}
\]

and substitute

\[
\tau = MJ_a^+ \dot{x} + \tau_a^{(12)} \tag{33}
\]

into the arm dynamics (12). In the resulting equation, we utilize (31) and the non-singularity of the mass matrix \( M \) to obtain

\[
\ddot{\theta} = J_a^+(\dot{x} - J_a \dot{\theta}),
\]

which clearly satisfies the equation

\[
\ddot{x} = J_a \dot{\theta} + J_a \dot{\theta}
\]

obtained from differentiating the kinematics (11). Equation (33) describes, in the Cartesian coordinates, correct arm dynamics because they are consistent with the kinematics.\(^3\)

\(^3\)Note that \( J_a^+ \) can be replaced with any right inverse \( A \) of \( J_a \) to yield arm dynamics consistent with (33).
Now we let $x_d = (\alpha T, \phi)^T$ be some desired trajectory and $x_e = x_d - x$ be the error. The controller in the pressing phase is derived from (33) by replacing $\bar{x}$ with

\[
\begin{pmatrix}
\lambda_x \\
\lambda_y \\
\lambda_\phi
\end{pmatrix} = \bar{x}_d + K_v[x] \bar{x}_e + K_p[x] x_e + K_i [x] \int x_e \, dt,
\]

where $K_v[x]$, $K_p[x]$, and $K_i[x]$ are all $3 \times 3$ positive-definite diagonal matrices. This results in the following closed loop system:

\[
M J_a^+ \left( \ddot{x}_e + K_v[x] \dot{x}_e + K_p[x] x_e + K_i [x] \int x_e \, dt \right) = 0.
\]

Multiplication of the above with $J_a M^{-1}$ and utilization of (31) yield the error dynamics:

\[
\ddot{x}_e + K_v[x] \dot{x}_e + K_p[x] x_e + K_i [x] \int x_e \, dt = 0,
\]

for which stability analysis is similar to that given in Appendix C-A for the controller (15) for a 2-DOF arm.

Starting in the touching phase, the knife is in contact with the cutting board. In this phase, the arm dynamics have an extra term than in (33) due to the contact force $f_y$ exerted by the cutting board:

\[
\tau = M J_a^+ \ddot{x}_e + \tau_a^{(32)} - J_c^T \left( \begin{array}{c} 0 \\ f_y \end{array} \right),
\]

where $J_c$, introduced in (20), is now a $2 \times n$ matrix. Impedance control takes a form similar to (22):

\[
\tau = M J_a^+ \begin{pmatrix} \lambda_x \\ \lambda_y \\ \lambda_\phi \end{pmatrix} + \tau_a^{(32)} - J_c^T \left( \begin{array}{c} 0 \\ f_y \end{array} \right).
\]

Stability analysis will be presented in Appendix D.

With more than two degrees of freedom, the arm can perform slicing without rotating the knife if needed. Lowest on the knife’s edge curve $\beta(u)$, the contact point $c$ satisfies (25), which induces the parameter value $u = \zeta(\theta)$ for some function $\zeta$. That this point lies on the cutting board introduces a constraint

\[
\beta_y(\zeta(\theta)) = 0.
\]

Under the new constraint, the independent variables are chosen to be $c_x$ and $\phi$, where $c_x = \beta_x(\zeta(\theta))$. Letting $y = (c_x, \phi)^T$, we have

\[
y = \begin{pmatrix} \dot{c}_x \\ \dot{\phi} \end{pmatrix} = \bar{J}_c \theta,
\]

where

\[
\bar{J}_c = \frac{\partial y}{\partial \theta}.
\]

Via a substitution of the expression of $\bar{\theta}$ obtained from differentiating (36), the system dynamics (19) for touching are transformed into

\[
\tau = M \bar{J}_c^+ \ddot{y} + \tau_c - J_c^T \left( \begin{array}{c} 0 \\ f_y \end{array} \right),
\]

where $\bar{J}_c^+$ is the pseudoinverse of $\bar{J}_c$ such that $\bar{J}_c \bar{J}_c^+ = I_2$, and $\tau_c$ is redefined from (29) below:

\[
\tau_c = (C - M \bar{J}_c^+ \bar{J}_c) \dot{\theta} + \tau_g - J_a^T \rho_a.
\]

To completely separate the object, during the slicing phase the knife maintains a certain level of force on the board to keep in direct contact. Let $c_{\text{xd}}(t)$ and $\phi_{\text{yd}}(t)$ be the desired trajectories of $c_x$ and $\phi$, respectively. We write $y_d = (c_{\text{xd}}, \phi_{\text{yd}})^T$. The following hybrid controller

\[
\tau = M \bar{J}_c^+ \begin{pmatrix} \lambda_x \\ \lambda_y \\ \lambda_\phi \end{pmatrix} + \tau_c^{(38)} - J_c^T \left( \begin{array}{c} 0 \\ f_d + k_f \int f_e \, dt \end{array} \right),
\]

where

\[
\begin{pmatrix} \lambda_x \\ \lambda_y \\ \lambda_\phi \end{pmatrix} = \bar{y}_d + K_v[c_x] \bar{y}_e + K_p[c_x] y_e + K_i[c] \int y_e \, dt,
\]

is proposed to realize the rate $c_{\text{xd}}(t)$ of separation while maintaining the knife’s constant orientation.

VII. DISCUSSION AND FUTURE WORK

This paper is about how to enable a robotic arm to perform a natural and smooth cutting action. We have sequenced the entire action into three phases (pressing, touching, and slicing), drawing inspirations from kitchen knife maneuvers by the human hand. Given the action’s complexity and the heavy presence of contacts (with the object and cutting board) engaged by the knife, a single policy such as position control is clearly inadequate for carrying out the task robustly. Instead, different control policies are employed sequentially to accommodate their own subgoals and specific contact constraints. As demonstrated in our experiments, these controls have smooth transitions to cut open a fruit/vegetable in about 2.5 s (see the submitted video).

Modeling based on fracture mechanics allows us to separate, from the F/T sensor reading, forces of different sources such as fracture, knife-material contact, and knife-board contact. In this work, we are able to estimate the knife-board contact force using modeled fracture and fractional forces, and consequently carry out impedance and hybrid controls during the touching and slicing phases.

An immediate extension will be to an object undergoing small deformations when it is being cut. Fracture and friction forces needed for cutting control, along with the
areas of fracture and contact and the object’s shape and strain energy, can be modeled using the finite element method (FEM) based on fracture mechanics. For cutting trajectory planning and efficient path planning, it is quite important to efficiently generate reliable force and shape predictions along a hypothesized trajectory. Meanwhile, a vision system can be employed to help reduce errors in shape tracking. We expect to compensate modeling inaccuracies with force sensing and improved knife control strategies. In the longer term, we would like to investigate cutting of objects with large deformations and viscosities. Modeling of strain energies and viscous forces can be quite important.

Another direction of extension is to have the kitchen knife held by a robotic hand driven by an arm rather than rigidly attached to the arm. The system will become more autonomous since the hand can pick up and regrasp the knife. There will be more dexterity because control of the knife is directly done by the hand. Cutting control will have to take into account issues such as higher degrees of freedom, compliance of contact between the knife’s handle and the hand, and (even possibly) finger gaits for adjusting a grasp on the knife. Involvement of a second robotic arm or hand to stabilize the object being cut will bring up the challenging issue of two-hand coordination, which expects the development of some control strategy to allow force interactions between the two hands.

At a higher level, we will look at how to plan trajectories to implement different knife skills including chop, slice, and dice. Realization of a composite skill such as dice, for instance, needs to tackle the subproblem of cutting in a non-vertical plane (which can be a direct extension to the current work).

**APPENDIX A**

**COMPUTATION OF FRACTURE AND FRICTIONAL FORCES FOR A TRANSLATING KNIFE**

When the knife keeps a constant orientation, all the points on its blade are moving at the same velocity \( \mathbf{v} \). Denote \( \mathbf{\dot{v}} = \mathbf{v}/||\mathbf{v}|| = (\mathbf{v}_x, \mathbf{v}_y)^T \). Calculation of fracture and frictional forces and torques can be simplified. The integral (7) now has a closed form:

\[
\mathbf{f}_C = \int_{u_1}^{u_2} \kappa \mathbf{\hat{v}} \cdot \left( - \frac{d\beta_y}{du} \frac{d\beta_x}{du} \right)^T \mathbf{\hat{v}} \, du
\]

\[
= \kappa \mathbf{\hat{v}} \cdot \left( \begin{array}{c} -\beta_y \\ \beta_x \end{array} \right)_{u_1}^{u_2} \mathbf{\hat{v}}. \tag{40}
\]

The torque is determined using (8):

\[
\tau_C = \int_{u_1}^{u_2} (\beta(u) - \mathbf{a}) \times \left( \kappa \mathbf{\hat{v}} \cdot \left( - \frac{d\beta_y}{du} \frac{d\beta_x}{du} \right)^T \mathbf{\hat{v}} \right) \, du
\]

\[
= \kappa \mathbf{\hat{v}} \cdot \int_{u_1}^{u_2} \left( (\beta(u) - \mathbf{a}) \times \mathbf{\hat{v}} \right) \left( - \frac{d\beta_y}{du} \frac{d\beta_x}{du} \right)^T \, du
\]

\[
= \kappa \mathbf{\hat{v}} \cdot \left( \begin{array}{c} -\beta_y \\ \beta_x \end{array} \right)_{u_1}^{u_2} (\mathbf{a} \times \mathbf{\hat{v}}) + \frac{1}{2} (\beta_y^2 \mathbf{\hat{v}}_x)_{u_1}^{u_2} (\beta_x^2 \mathbf{\hat{v}}_y)_{u_1}^{u_2} \mathbf{\hat{v}}.
\]

\[
\tau_C = \int_{u_1}^{u_2} \left( \beta_x \beta_y \mathbf{\hat{v}}_y \right) \, du. \tag{41}
\]

Under Green’s theorem, the area \( A \) of the knife-material contact can be calculated along its boundary \( \partial \Omega \) as follows:

\[
A = \int_{\partial \Omega} x \, dy = \int_{\partial \Omega} \mathbf{\hat{a}} \cdot \mathbf{n} \, ds
\]

\[
A = \int_{u_1}^{u_2} \beta_x \, d\beta_y + \int_{r_3}^{r_4} \sigma_x \, d\sigma_y + \int_{q_2}^{q_1} \gamma_x \, d\gamma_y \tag{42}
\]

\[
+ \int_{r_4} \sigma_x \, d\sigma_y.
\]

The frictional force (9) and its generated torque (10) are

\[
\mathbf{f}_F = -2\mu P \mathbf{\hat{a}} \mathbf{\hat{v}}, \tag{43}
\]

\[
\tau_F = -2\mu P \int_{\Omega} \left( \begin{array}{c} x \\ y \end{array} \right) \times \mathbf{\hat{v}} \, dxdy
\]

\[
= 2\mu P \mathbf{a} \times \mathbf{\hat{v}} - 2\mu P \int_{\Omega} \left( \begin{array}{c} x \\ y \end{array} \right) \times \mathbf{\hat{v}} \, dxdy
\]

\[
= 2\mu P \mathbf{a} \times \mathbf{\hat{v}} - 2\mu P \int_{\Omega} (xv_y - yv_x) \, dxdy
\]

\[
= 2\mu P \mathbf{a} \times \mathbf{\hat{v}} - 2\mu P \int_{\partial \Omega} \frac{x^2}{2} v_y - xyv_x \, dy. \tag{44}
\]

The integrals (40)–(44) have closed forms when the curves \( \beta, \gamma, \) and \( \delta \) for the knife’s edge and spine and the object’s cross section are parameterized using polynomials. Such parameterizations are easy to generate, as conducted in our experiments in Section V.

**APPENDIX B**

**JACOBIAN EVALUATION**

In this appendix, we evaluate the Jacobian \( \mathbf{J}_c \) defined in (26) and its derivative \( \mathbf{J}_c^\prime \) related to the knife-board contact \( \mathbf{c} \). They are used for control in the slicing phase. Writing \( \mathbf{c} = (\theta_1, \theta_2, u)^T \), so \( \beta \) is a function of \( \mathbf{c} \), we differentiate (23)–(25) with respect to \( \mathbf{c}_x^\prime \):

\[
\begin{pmatrix}
\frac{d\beta}{dc_x} \\
\frac{d}{dc_x} (\frac{\partial \beta_y}{\partial u})
\end{pmatrix}
\]

\[
= B \frac{d\mathbf{c}}{dc_x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{45}
\]

\[
= B \frac{d\mathbf{c}}{dc_x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{45}
\]

\[
= B \frac{d\mathbf{c}}{dc_x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{45}
\]
where the $3 \times 3$ matrix $B$ is
\[
B = \begin{pmatrix}
\frac{\partial \beta}{\partial \xi} \\
\frac{\partial \beta}{\partial \epsilon}
\end{pmatrix}.
\]
(46)

Solving the linear system (45), we obtain $d\xi/dc_x$ and thus $t_c$ since
\[
d\xi/dc_x = \left(\begin{array}{c} t_c \\
du/dc_x \\
\end{array} \right).
\]
(47)

Next, we evaluate the derivative
\[
t_c = \left(\frac{d^2 \theta_1}{dc_x^2}, \frac{d^2 \theta_2}{dc_x^2}\right)^T.
\]
(48)

The time derivative of $c_x$ is from differentiating (23):
\[
\dot{c}_x = \frac{\partial c_x}{\partial \xi} \hat{\xi},
\]

Among the components of $\hat{\xi}$, $\dot{\theta}_1$ and $\dot{\theta}_2$ are read from the controller of the robotic arm, and $\dot{u}$ is obtained from differentiating (25):
\[
\dot{u} = -\left(\frac{\partial^2 \beta_y}{\partial \theta_1 \partial \theta_1} + \frac{\partial^2 \beta_y}{\partial \theta_2 \partial \theta_1}\right) / \partial^2 \beta_y.
\]

What remain to be evaluated in (48) are the two partial derivatives $d^2 \theta_1/dc_x^2$ and $d^2 \theta_2/dc_x^2$. It suffices to obtain $d^2 \xi/dc_x^2$. Differentiation of (45) with respect to $c_x$ yields
\[
dB/dc_x + B d^2 \xi/dc_x = 0,
\]
from which we have
\[
d^2 \xi/dc_x^2 = -B^{-1} dB/dc_x.
\]

Of the three factors in the right hand side of the above equation, the matrix $B = (b_{ij})$ by (46) and derivative $d\xi/dc_x$ by (47) are both available. The final step is to evaluate the $3 \times 3$ matrix $dB/dc_x = (db_{ij}/dc_x)$, where
\[
db_{ij}/dc_x = \frac{dB}{dc_x} / \partial \xi dc_x.
\]

The partial derivatives $db_{ij}/dc_x$ are directly from differentiating (46).

**Appendix C**

**Stability Proofs**

In this appendix, stability of the controller in each of the three phases are proved for the 2-DOF arm.

**A. Pressing**

In the pressing phase, the position controller (15) is applied to let the robot follow a desired cutting path. The error dynamics can be obtained by equating (13) and (15), yielding
\[
M \ddot{J}_a^{-1} \left(\bar{a}_e + K_{v|a} \hat{a}_e + K_{p|a} a_e + K_{i|a} \int \alpha_e dt\right) = 0.
\]

Multiply $\ddot{J}_a^{-1} = (\ddot{J}_a^{-1})^T$ with both sides of (49), since the left side $\ddot{J}_a^{-1} M \ddot{J}_a^{-1}$ is positive-definite and the right side is a zero vector, leading to
\[
\ddot{a}_e + K_{v|a} \hat{a}_e + K_{p|a} a_e + K_{i|a} \int \alpha_e dt = 0.
\]

The gain matrices $K_{v|a}$, $K_{p|a}$, and $K_{i|a}$ are constant, diagonal, and positive definite. The error equation (50) can be rewritten as a linear time invariant (LTI) system
\[
\dot{z} = Az,
\]
with the state
\[
z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \int \alpha_e dt \\ a_e \\ \hat{a}_e \end{pmatrix}
\]

and the coefficient matrix
\[
A = \begin{pmatrix} 0 & I_2 & 0 \\ 0 & 0 & I_2 \\ -K_{i|a} & -K_{p|a} & -K_{v|a} \end{pmatrix}.
\]

An eigenvalue $\lambda$ of $A$ satisfies
\[
\lambda \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = Az = \begin{pmatrix} z_2 \\ z_3 \end{pmatrix}.
\]

It follows that
\[
z_1^T \lambda^3 z_1 = z_1^T \lambda^2 z_2 = z_1^T \lambda z_3
\]

\[
= -z_1^T Q_1(a_z + K_{p|a} z_2 + K_{v|a} z_3)
\]

\[
= -z_1^T K_{i|a} z_1 - \lambda z_1^T K_{p|a} z_1 - \lambda^2 z_1^T K_{v|a} z_1.
\]

Introducing $\dot{z}_1 = z_1/\|z_1\|$, the above equation reduces to
\[
\lambda^3 + \alpha_2 \lambda^2 + \alpha_1 \lambda + \alpha_0 = 0,
\]

where $\alpha_0 = \hat{z}_1^T K_{i|a} \hat{z}_1$, $\alpha_1 = \hat{z}_1^T K_{p|a} \hat{z}_1$, and $\alpha_2 = \hat{z}_1^T K_{v|a} \hat{z}_1$ are all positive scalars since the three gain matrices are positive definite.

Stability of the LTI system is achieved if the real part of $\lambda$ is negative [41, pp.177]. This condition holds under $\alpha_2 < \alpha_1 > \alpha_0$ [42, pp.394]. The inequality is easy to realize by choosing the diagonal elements of the three gain matrices properly. This is because $\alpha_1$ and $\alpha_2$ are
bounded from below by the smallest diagonal entries (eigenvalues) of \( K_{p\alpha} \) and \( K_{v\alpha} \), and \( \alpha_0 \) is bounded from above by the largest diagonal entry of \( K_{i\alpha} \).

B. Touching

In the touching phase, the impedance controller (22) is designed to make the robot decelerate quickly upon contact with the cutting board. The closed-loop system can be obtained by equating (21) and (22):

\[
MJ^{-1}\begin{pmatrix}
\hat{a}_{xe} + k_{i\alpha}a_{xe} + k_{p\alpha}a_{xe} + k_{i\alpha} \int a_{xe} \, dt \\
\hat{a}_{ye} + k_r a_{ye} + d_r \dot{a}_{ye} + f_y \\
\end{pmatrix} = 0.
\]

Multiply \( J_a^{-T} \) with both sides of (51), we can obtain the \( x \)-direction position control error dynamics:

\[
\dot{a}_{xe} + k_{i\alpha}a_{xe} + k_{p\alpha}a_{xe} + k_{i\alpha} \int a_{xe} \, dt = 0,
\]

and the \( y \)-direction impedance control error dynamics:

\[
b_r \ddot{a}_{ye} + k_r a_{ye} + d_r \dot{a}_{ye} + f_y = 0.
\]

Equation (52) can be rewritten as an LTI system with the state chosen as

\[
\dot{z} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \int a_{xe} \, dt \\ a_{xe} \\ \dot{a}_{xe} \end{pmatrix}.
\]

This system \( \dot{z} = Az \) expands into

\[
\begin{pmatrix} a_{xe} \\ \dot{a}_{xe} \\ \ddot{a}_{xe} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_{i\alpha} & -k_{p\alpha} & -k_{i\alpha} \end{pmatrix} \begin{pmatrix} \int a_{xe} \, dt \\ a_{xe} \\ \dot{a}_{xe} \end{pmatrix}.
\]

Stability is achieved by positive gains \( k_{p\alpha}, k_{v\alpha}, \) and \( k_{i\alpha} \) satisfying \( k_{p\alpha} > k_{i\alpha} \).

Equation (53) admits the following state space expression:

\[
\dot{a}_{ye} = \frac{1}{b_r} \begin{pmatrix} 0 & b_r \\ -k_r & -d_r \end{pmatrix} \begin{pmatrix} a_{ye} \\ \dot{a}_{ye} \end{pmatrix} + \frac{1}{b_r} \begin{pmatrix} 0 \\ 1 \end{pmatrix} f_y.
\]

Since the contact force \( f_y \) is bounded, the above system’s bounded input bounded output (BIBO) stability is guaranteed if we ensure \( b_r > 0, k_r > 0 \) and \( d_r > 0 \). The force \( f_y \) has its magnitude related to the position error \( a_{ye} \). Both \( f_y \) and \( a_{ye} \) will converge to constant value as \( \dot{a}_{ye} \) converges to 0.

C. Slicing

In the slicing phase, hybrid force/position control is applied to regulate the cutting velocity in the horizontal direction as well as the contact force in the vertical direction. The closed-loop system equation obtained from the dynamics (28) and the control (30) has the form

\[
M \dot{c} + \begin{pmatrix} \dot{c}_{xe} \\ \dot{c}_{ye} \end{pmatrix} + k_{p\alpha}c_{xe} + k_{p\alpha}c_{ye} + k_{i\alpha} \int c_{xe} \, dt = 0. \tag{54}
\]

Next, we show that the last term in (54) can be eliminated via left multiplication with \( \dot{c}_0 \). Equivalently, we need to establish that \((0,1)J_c \dot{c}_0 = 0 \). The reasoning proceeds as follows:

\[
(0,1)J_c \dot{c}_0 = \begin{pmatrix} \partial c_x \\ \partial c_y \\ \partial \theta_1 \\ \partial \theta_2 \end{pmatrix} \begin{pmatrix} \partial \theta_1 \\ \partial \theta_2 \end{pmatrix} = \partial c_u \partial \theta_1 + \partial c_u \partial \theta_2 = 0.
\]

Now, we differentiate (24) with respect to \( u \), treating \( \theta_1 \) and \( \theta_2 \) as functions of \( u \) given by (24) and (25):

\[
\frac{\partial \beta_y}{\partial u} + \frac{\partial \beta_y}{\partial \theta_1} \frac{\partial \theta_1}{\partial u} + \frac{\partial \beta_y}{\partial \theta_2} \frac{\partial \theta_2}{\partial u} = 0. \tag{56}
\]

Because \( c = \beta(u) \), (55) and (56) together imply that \((0,1)J_c \dot{c}_0 = 0 \), which in turn implies

\[
J_c^T J_c \begin{pmatrix} 0 \\ f_x + k_{f\beta} \int f_x \, dt \end{pmatrix} = 0.
\]

The above equation allows us to eliminate the force control term from (54) via left multiplication with \( J_c^T \), resulting in the following error dynamics:

\[
\dot{c}_{xe} + k_{i\alpha}c_{xe} + k_{p\alpha}c_{xe} + k_{i\alpha} \int c_{xe} \, dt = 0, \tag{57}
\]

for which stability analysis is similar to that of \( x \)-direction position control in Appendix C-B.

Substitution of (57) back into (54) generates

\[
f_x + k_{f\beta} \int f_x \, dt = 0.
\]

This is a first order LTI system whose stability can be ensured by a positive value of the gain \( k_{f\beta} \).
Meanwhile, we substitute (61) into (59) to obtain $f_e + k_{fi} \int f_e \, dt = 0$, which is stable with a positive gain $k_{fi}$.

APPENDIX D
STABILITY ANALYSIS FOR CUTTING WITH AN n-DOF ARM

In this appendix, we will establish stabilities of the controllers (35) and (39) employed during the touching and slicing phases of cutting with an $n$-DOF arm.

Let us start with the first controller (35). The closed-loop system equation follows from equating (34) and (35):

$$MJ_a^+ \begin{bmatrix} \ddot{a}_{xc} + k_{\bar{v}|a}\dot{a}_{xc} + k_{p|a}a_{xc} + k_{i|a} \int a_{xc} \, dt \\ \ddot{a}_{yc} + (k_r a_{yc} + a_r \dot{a}_{yc} + f_y) / b_r \\ \ddot{\phi}_{xc} + k_{\bar{v}|a}\dot{\phi}_{xc} + k_{p|a}\phi_{xc} + k_{i|a} \int \phi_{xc} \, dt \end{bmatrix} = 0.$$  

(58)

Multiplication of $J_a^{+T}$ with both sides of (58), given that $J_a^{+T}MJ_a^+$ is positive-definite, yielding

$$\ddot{a}_{xc} + k_{\bar{v}|a}\dot{a}_{xc} + k_{p|a}a_{xc} + k_{i|a} \int a_{xc} \, dt,$$

$$b_r \ddot{a}_{yc} + k_r a_{yc} + a_r \dot{a}_{yc} + f_y = 0,$$

$$\ddot{\phi}_{xc} + k_{\bar{v}|a}\dot{\phi}_{xc} + k_{p|a}\phi_{xc} + k_{i|a} \int \phi_{xc} \, dt = 0.$$  

Stabilities of the three equations above are guaranteed by positive gains $k_{\bar{v}|a}$, $k_{p|a}$ and $k_{i|a}$ satisfying $k_{p|a}k_{\bar{v}|a} > k_{i|a}$ (see Appendix C-B), and positive $b_r$, $a_r$, and $k_r$ (see Appendix C-B).

For the slicing phase, the closed-loop system equation is obtained by subtracting (37) from (39):

$$MJ_e^+ \begin{bmatrix} \dot{y}_e + K_{\bar{v}|e}\dot{y}_e + K_{p|e}y_e + K_{i|e} \int y_e \, dt \\ 0 \end{bmatrix} = -J_c^{+T} \begin{bmatrix} f_e + k_{fi} \int f_e \, dt \end{bmatrix} = 0.$$  

(59)

In the above equation, $J_c^{+T}(0, 1)^T$ is an $n \times 1$ vector. Let $Z$ be a $2 \times n$ matrix of full rank such that $ZJ_c^{+T}(0, 1)^T = 0$. Left multiplying $Z$ with both sides of (59), we have

$$ZMJ_e^+ \begin{bmatrix} \dot{y}_e + K_{\bar{v}|e}\dot{y}_e + K_{p|e}y_e + K_{i|e} \int y_e \, dt \end{bmatrix} = 0.$$  

(60)

If the matrix $Z_e$ is not singular, the rank of the $2 \times 2$ matrix $ZMJ_e^+$ is 2. Then the error dynamics are extracted from (60):

$$\dot{y}_e + K_{\bar{v}|e}\dot{y}_e + K_{p|e}y_e + K_{i|e} \int y_e \, dt = 0.$$  

(61)

Stability can be obtained by selecting appropriate controller gains as shown for (50) in Appendix C-A. Meanwhile, we substitute (61) into (59) to obtain $f_e + k_{fi} \int f_e \, dt = 0$, which is stable with a positive gain $k_{fi}$.

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REFERENCES


